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Abstract. Let G be a finite group and let k be a sufficiently large finite field. Let $\mathcal{R}(G)$ denote the character ring of G (i.e. the Grothendieck ring of the category of $\mathbb{C}G$-modules). We study the structure and the representations of the commutative algebra $k \otimes_{\mathbb{Z}} \mathcal{R}(G)$.

Let G be a finite group. We denote by $\mathcal{R}(G)$ the *Grothendieck ring* of the category of $\mathbb{C}G$-modules (it is usually called the *character ring* of G). It is a natural question to try to recover properties of G from the knowledge of $\mathcal{R}(G)$. It is clear that two finite groups having the same character table have the same Grothendieck rings and it is a Theorem of Saksonov [S] that the converse also holds. So the problem is reduced to an intensively studied question in character theory: recover properties of the group through properties of its character table.

In this paper, we study the k-algebra $k\mathcal{R}(G) = k \otimes_{\mathbb{Z}} \mathcal{R}(G)$, where k is a splitting field for G of positive characteristic p. It is clear that the knowledge of $k\mathcal{R}(G)$ is a much weaker information than the knowledge of $\mathcal{R}(G)$. The aim of this paper is to gather results on the representation theory of the algebra $k\mathcal{R}(G)$: although most of the results are certainy well-known, we have not found any general treatment of these questions. The blocks of $k\mathcal{R}(G)$ are local algebras which are parametrized by conjugacy classes of p-regular elements of G. So the simple $k\mathcal{R}(G)$-modules are parametrized by conjugacy classes of p-regular elements of G. Moreover, the dimension of the projective cover of the simple module associated to the conjugacy class of the p-regular element $g \in G$ is equal to the number of conjugacy classes of p-elements in the centralizer $C_G(g)$. We also prove that the radical of $k\mathcal{R}(G)$ is the kernel of the decomposition map $k\mathcal{R}(G) \to k \otimes_{\mathbb{Z}} \mathcal{R}(kG)$, where $\mathcal{R}(kG)$ is the Grothendieck ring of the category of kG-modules (i.e. the ring of virtual Brauer characters of G).

We prove that the block of $k\mathcal{R}(G)$ associated to the p'-element g is isomorphic to the block of $k\mathcal{R}(C_G(g))$ associated to 1 (such a block is called the *principal block*). This shows that the study of blocks of $k\mathcal{R}(G)$ is reduced to the study of principal blocks. We also show that the principal block of $k\mathcal{R}(G)$ is isomorphic to the principal block of $k\mathcal{R}(H)$

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whenever H is a subgroup of p'-index which controls the fusion of p-elements or whenever H is the quotient of G by a normal p'-subgroup.

We also introduce several numerical invariants (Loewy length, dimension of Ext-groups) that are partly related to the structure of G. These numerical invariants are computed completely whenever G is the symmetric group $\mathfrak{S}_n$ (this relies on previous work of the author: the descending Loewy series of $k\mathcal{R}(\mathfrak{S}_n)$ was entirely computed in [B]) or G is a dihedral group and $p = 2$. We also provide tables for these invariants for small groups (alternating groups $\mathfrak{A}_n$ with $n \leqslant 12$, some small simple groups, groups $PSL(2,q)$ with q a prime power $\leqslant 27$, exceptional finite Coxeter groups).

Notation - Let $\mathcal{O}$ be a Dedekind domain of characteristic zero, let $\mathfrak{p}$ be a maximal ideal of $\mathcal{O}$, let K be the fraction field of $\mathcal{O}$ and let $k = \mathcal{O}/\mathfrak{p}$. Let $\mathcal{O}_\mathfrak{p}$ be the localization of $\mathcal{O}$ at $\mathfrak{p}$: then $k = \mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p}$. If $x \in \mathcal{O}_\mathfrak{p}$, we denote by $\bar{x}$ its image in $\mathcal{O}_\mathfrak{p}/\mathfrak{p}\mathcal{O}_\mathfrak{p} = k$. Throughout this paper, we assume that k has characteristic $p > 0$ and that K and k are splitting fields for all the finite groups involved in this paper. If n is a non-zero natural number, $n_{p'}$ denotes the largest divisor of n prime to p and we set $n_p = n/n_{p'}$.

If F is a field and if A is a finite dimensional F-algebra, we denote by $\mathcal{R}(A)$ its Grothendieck group. If M is an A-module, the radical of M is denoted by $\operatorname{Rad} M$ and the class of M in $\mathcal{R}(A)$ is denoted by $[M]$. If S is a simple A-module, we denote by $[M : S]$ the multiplicity of S as a chief factor of a Jordan-Hölder series of M. The set of irreducible characters of A is denoted by $\operatorname{Irr} A$.

We fix all along this paper a finite group G. For simplification, we set $\mathcal{R}(G) = \mathcal{R}(KG)$ and $\operatorname{Irr} G = \operatorname{Irr} KG$ (recall that K is a splitting field for G). The abelian group $\mathcal{R}(G)$ is endowed with a structure of ring induced by the tensor product. If $\chi \in \mathcal{R}(G)$, we denote by χ^* its dual (as a class function on G, we have $\chi^*(g) = \chi(g^{-1})$ for any $g \in G$). If R is any commutative ring, we denote by $\operatorname{Class}_R(G)$ the space of class functions $G \to R$ and we set $R\mathcal{R}(G) = R \otimes_\mathbb{Z} \mathcal{R}(G)$. If X is a subset of G, we denote by $1_X^R : G \to R$ the characteristic function of X. If R is a subring of K, then we simply write $1_X = 1_X^R$. Note that 1_G is the trivial character of G. If $f,\ f' \in \operatorname{Class}_K(G)$, we set

$$\langle f, f' \rangle_G = \frac{1}{|G|} \sum_{g \in G} f(g^{-1}) f'(g).$$

Then $\operatorname{Irr} G$ is an orthonormal basis of $\operatorname{Class}_K(G)$. We shall identify $\mathcal{R}(G)$ with the sub-$\mathbb{Z}$-module (or sub-$\mathbb{Z}$-algebra) of $\operatorname{Class}_K(G)$ generated by $\operatorname{Irr} G$, and $K\mathcal{R}(G)$ with $\operatorname{Class}_K(G)$. If $f \in \mathcal{O}_\mathfrak{p}\mathcal{R}(G)$, we denote by $\bar{f}$ its image in $k\mathcal{R}(G)$.

If g and h are two elements of G, we write $g \sim h$ (or $g \sim_G h$ if we need to emphasize the group) if they are conjugate in G. We denote by g_p (resp. $g_{p'}$) the p-part (resp. the p'-part) of g. If X is a subset of G, we set $X_{p'} = \{g_{p'} \mid g \in X\}$ and $X_p = \{g_p \mid g \in X\}$. If moreover X is closed under conjugacy, the set of conjugacy classes contained in X is denoted by $X/\!\sim$. In this case, $1_X^R \in \operatorname{Class}_R(G)$. The centre of G is denoted by $Z(G)$.

1. Preliminaries

1.A. **Symmetrizing form.** Let

$$\begin{array}{rccc} \tau_G : & \mathcal{R}(G) & \longrightarrow & \mathbb{Z} \\ & \chi & \longmapsto & \langle \chi, 1_G \rangle_G \end{array}$$

denote the canonical symmetrizing form on $\mathcal{R}(G)$. The dual basis of $\operatorname{Irr} G$ is $(\chi^*)_{\chi \in \operatorname{Irr} G}$. It is then readily seen that $(\mathcal{R}(G), \operatorname{Irr} G)$ is a *based ring* (in the sense of Lusztig [L, Page 236]).

If R is any ring, we denote by $\tau_G^R : R\mathcal{R}(G) \to R$ the symmetrizing form $\operatorname{Id}_R \otimes_{\mathbb{Z}} \tau_G$.

1.B. **Translation by the centre.** If $\chi \in \operatorname{Irr} G$, we denote by $\omega_\chi : Z(G) \to \mathcal{O}^\times$ the linear character such that $\chi(zg) = \omega_\chi(z)\chi(g)$ for all $z \in Z(G)$ and $g \in G$. If $z \in Z(G)$, we denote by $t_z : K\mathcal{R}(G) \to K\mathcal{R}(G)$ the linear map defined by $(t_z f)(g) = f(zg)$ for all $f \in K\mathcal{R}(G)$ and $g \in G$. It is clear that $t_{zz'} = t_z \circ t_{z'}$ for all z, $z' \in Z(G)$ and that t_z is an automorphism of algebra. Moreover,

$$t_z \chi = \omega_\chi(z)\chi$$

for every $\chi \in \operatorname{Irr} G$. Therefore, t_z is an isometry which stabilizes $\mathcal{O}\mathcal{R}(G)$. If R is a subring of K such that $\mathcal{O} \subset R \subset K$, we still denote by $t_z : R\mathcal{R}(G) \to R\mathcal{R}(G)$ the restriction of t_z. Let $\bar{t}_z = \operatorname{Id}_k \otimes_{\mathcal{O}} t_z : k\mathcal{R}(G) \to k\mathcal{R}(G)$. This is again an automorphism of k-algebra. If z is a p-element, then $\bar{t}_z = \operatorname{Id}_{k\mathcal{R}(G)}$.

1.C. **Restriction.** If $\pi : H \to G$ is a morphism of groups, then the *restriction through* π induces a morphism of rings $\operatorname{Res}_\pi : \mathcal{R}(G) \to \mathcal{R}(H)$. If R is a subring of K, we still denote by $\operatorname{Res}_\pi : R\mathcal{R}(G) \to R\mathcal{R}(H)$ the morphism $\operatorname{Id}_R \otimes_{\mathbb{Z}} \operatorname{Res}_\pi$. We denote by $\overline{\operatorname{Res}}_\pi$: $k\mathcal{R}(G) \to k\mathcal{R}(H)$ the reduction modulo $\mathfrak{p}$ of $\operatorname{Res}_\pi : \mathcal{O}\mathcal{R}(G) \to \mathcal{O}\mathcal{R}(H)$. Recall that, if H is a subgroup of G and π is the canonical injection, then Res_π is just Res_H^G. In this case, $\overline{\operatorname{Res}}_\pi$ will be denoted by $\overline{\operatorname{Res}}_H^G$. Note the following fact:

(1.1) *If π is surjective, then $\overline{\operatorname{Res}}_\pi$ is injective.*

Proof of 1.1. Indeed, if π is surjective, then $\operatorname{Res}_\pi : \mathcal{R}(G) \to \mathcal{R}(H)$ is injective and its image is a direct summand of $\mathcal{R}(H)$. ∎

1.D. **Radical.** First, note that, since $k\mathcal{R}(G)$ is commutative, we have

(1.2) $\operatorname{Rad} k\mathcal{R}(G)$ *is the ideal of nilpotent elements of* $k\mathcal{R}(G)$.

So, if $\pi : H \to G$ is a morphism of finite groups, then

(1.3) $$\overline{\operatorname{Res}}_\pi\big(\operatorname{Rad} k\mathcal{R}(G)\big) \subset \operatorname{Rad} k\mathcal{R}(H).$$

The *Loewy length* of the algebra $k\mathcal{R}(G)$ is defined as the smallest natural number n such that $\big(\operatorname{Rad} k\mathcal{R}(G)\big)^n = 0$. We denote it by $\ell_p(G)$. By 1.1 and 1.3, we have:

(1.4) *If π is surjective, then $\ell_p(G) \leqslant \ell_p(H)$.*

2. MODULES FOR $K\mathcal{R}(G)$ AND $k\mathcal{R}(G)$

2.A. **Semisimplicity.** Recall that $K\mathcal{R}(G)$ is identified with the algebra of class functions on G. If $C \in G/\!\sim$ and $f \in K\mathcal{R}(G)$, we denote by $f(C)$ the constant value of f on C. We now define $\operatorname{ev}_C : K\mathcal{R}(G) \to K$, $f \mapsto f(C)$. It is a morphism of K-algebras. In other words, it is an irreducible representation (or character) of $K\mathcal{R}(G)$. We denote by $\mathcal{D}_C$ the corresponding simple $K\mathcal{R}(G)$-module ($\dim_K \mathcal{D}_C = 1$ and an element $f \in K\mathcal{R}(G)$ acts on $\mathcal{D}_C$ by multiplication by $\operatorname{ev}_C(f) = f(C)$). Now, 1_C is a primitive idempotent of $K\mathcal{R}(G)$ and it is easily checked that

(2.1) $$K\mathcal{R}(G)1_C \simeq \mathcal{D}_C.$$

Recall that

(2.2) $$1_C = \frac{|C|}{|G|} \sum_{\chi \in \operatorname{Irr} G} \chi(C^{-1})\chi$$

and

(2.3) $$\sum_{C \in G/\sim} 1_C = 1_G.$$

Therefore:

Proposition 2.4. *We have:*

(a) *$(\mathcal{D}_C)_{C \in G/\sim}$ is a family of representatives of isomorphy classes of simple $K\mathcal{R}(G)$-modules.*
(b) *$\operatorname{Irr} K\mathcal{R}(G) = \{\operatorname{ev}_C \mid C \in G/\!\sim\}$.*
(c) *$K\mathcal{R}(G)$ is split semisimple.*

We conclude this section by the computation of the Schur elements (see [GP, 7.2] for the definition) associated to each irreducible character of $K\mathcal{R}(G)$. Since

(2.5) $$\tau_G^K = \sum_{C \in G/\sim} \frac{|C|}{|G|} \operatorname{ev}_C,$$

we have by [GP, Theorem 7.2.6]:

Corollary 2.6. *Let $C \in G/\sim$. Then the Schur element associated with the irreducible character* ev_C *is* $\dfrac{|G|}{|C|}$.

REMARK 2.7 - If $z \in Z(G)$, then t_z induces an isomorphism of algebras $K\mathcal{R}(G)1_C \simeq K\mathcal{R}(G)1_{z^{-1}C}$. □

REMARK 2.8 - If $f \in K\mathcal{R}(G)$, then $f = \sum_{C \in G/\sim} f(C)1_C$. □

EXAMPLE 2.9 - The map ev_1 will sometimes be denoted by deg, since it sends a character to its degree. □

2.B. **Decomposition map.** Let $d_{\mathfrak{p}} : \mathcal{R}(G) \to \mathcal{R}(kG)$ denote the decomposition map. If R is any commutative ring, we denote by $d_{\mathfrak{p}}^R : R\mathcal{R}(G) \to R\mathcal{R}(kG)$ the induced map. Note that $\mathcal{R}(kG)$ is also a ring (for the multiplication given by tensor product) and that $d_{\mathfrak{p}}$ is a morphism of ring. Also, by [CR, Corollary 18.14],

(2.10) $d_{\mathfrak{p}}$ *is surjective.*

Since $\mathrm{Irr}(kG)$ is a linearly independent family of class functions $G \to k$ (see [CR, Theorem 17.4]), the map $\chi : k\mathcal{R}(kG) \to \mathrm{Class}_k(G)$ that sends the class of a kG-module to its character is (well-defined and) injective. This is a morphism of k-algebras.

Now, if C is a conjugacy class of p-regular elements (i.e. $C \in G_{p'}/\sim$), we define

$$\mathcal{S}_{p'}(C) = \{g \in G \mid g_{p'} \in C\}$$

(for instance, $\mathcal{S}_{p'}(1) = G_p$). Then $\mathcal{S}_{p'}(C)$ is called the *p'-section* of C: this is a union of conjugacy classes of G. Let $\mathrm{Class}_k^{p'}(G)$ be the space of class functions $G \to k$ which are constant on p'-sections. Then, by [CR, Lemma 17.8], $\mathrm{Irr}(kG) \subset \mathrm{Class}_k^{p'}(G)$, so the image of χ is contained in $\mathrm{Class}_k^{p'}(G)$. But, χ is injective, $|\mathrm{Irr}(kG)| = |G_{p'}/\sim|$ (see [CR, Corollary 17.11]) and $\dim_k \mathrm{Class}_k^{p'}(G) = |G_{p'}/\sim|$. Therefore, we can identify, through χ, the k-algebras $k\mathcal{R}(kG)$ and $\mathrm{Class}_k^{p'}(G)$. In particular,

(2.11) $k\mathcal{R}(kG)$ *is split semisimple.*

2.C. **Simple $k\mathcal{R}(G)$-modules.** If $C \in G/\sim$, we still denote by $\mathrm{ev}_C : \mathcal{O}\mathcal{R}(G) \to \mathcal{O}$ the restriction of ev_C and we denote by $\overline{\mathrm{ev}}_C : k\mathcal{R}(G) \to k$ the reduction modulo $\mathfrak{p}$ of ev_C. It is easily checked that $\overline{\mathrm{ev}}_C$ factorizes through the decomposition map $d_{\mathfrak{p}}$. Indeed, if $\mathrm{ev}_C^k : k\mathcal{R}(kG) \to k$ denote the evaluation at C (recall that $k\mathcal{R}(kG)$ is identified, via the map χ of the previous subsection, to $\mathrm{Class}_k^{p'}(G)$), then

(2.12) $\overline{\mathrm{ev}}_C = \mathrm{ev}_C^k \circ d_{\mathfrak{p}}^k.$

Let $\bar{\mathcal{D}}_C$ be the corresponding simple $k\mathcal{R}(G)$-module. Let $\delta_{\mathfrak{p}} : \mathcal{R}(K\mathcal{R}(G)) \to \mathcal{R}(k\mathcal{R}(G))$ denote the decomposition map (see [GP, 7.4] for the definition). Then

(2.13) $\delta_{\mathfrak{p}}[\mathcal{D}_C] = [\bar{\mathcal{D}}_C].$

The following facts are well-known:

Proposition 2.14. *Let C, $C' \in G/\!\sim$. Then $\bar{\mathcal{D}}_C \simeq \bar{\mathcal{D}}_{C'}$ if and only if $C_{p'} = C'_{p'}$.*

Proof. The "if" part follows from the following classical fact [CR, Proposition 17.5 (ii) and (iv) and Lemma 17.8]: if $\chi \in \mathcal{R}(G)$ and if $g \in G$, then

$$\chi(g) \equiv \chi(g_{p'}) \mod \mathfrak{p}.$$

The "only if" part follows from 2.12 and from the surjectivity of the decomposition map $d_\mathfrak{p}$. □

Corollary 2.15. *We have:*

(a) $(\bar{\mathcal{D}}_C)_{C\in G_{p'}/\sim}$ *is a family of representatives of isomorphy classes of simple $k\mathcal{R}(G)$-modules.*
(b) $\operatorname{Irr} k\mathcal{R}(G) = \{\overline{\mathrm{ev}}_C \mid C \in G_{p'}/\!\sim\}$.
(c) $\operatorname{Rad} k\mathcal{R}(G) = \operatorname{Ker} d_\mathfrak{p}^k$.
(d) $k\mathcal{R}(G)$ *is split.*

Proof. (a) follows from 2.13 and from the fact that the isomorphy class of any simple $k\mathcal{R}(G)$-modules must occur in some $\delta_\mathfrak{p}[S]$, where S is a simple $K\mathcal{R}(G)$-module. (b) follows from (a). (c) and (d) follow from (a), (b), 2.12 and 2.11. □

Corollary 2.16. $\dim_k \operatorname{Rad}(k\mathcal{R}(G)) = |G/\!\sim| - |G_{p'}/\!\sim|$.

Corollary 2.17. *$k\mathcal{R}(G)$ is semisimple if and only if p does not divide $|G|$.*

Example 2.18 - Since ev_1 is also denoted by deg, we shall sometimes denote by $\overline{\deg}$ the morphism $\overline{\mathrm{ev}}_1$. If G is a p-group, then Corollary 2.15 shows that $\operatorname{Rad} k\mathcal{R}(G) = \operatorname{Ker}(\overline{\deg})$. In this case, if $1, \lambda_1, \dots, \lambda_r$ denote the linear characters of G and $\chi_1, \dots, \chi_s$ denote the non-linear irreducible characters of G, then $(\overline{\lambda}_1 - 1, \dots, \overline{\lambda}_r - 1, \bar{\chi}_1, \bar{\chi}_s)$ is a k-basis of $\operatorname{Rad} k\mathcal{R}(G)$. □

2.D. **Projective modules.** We now fix a conjugacy class C of p-regular elements (i.e. $C \in G_{p'}/\!\sim$). Let

$$e_C = 1_{\mathcal{S}_{p'}(C)} = \sum_{D\in\mathcal{S}_{p'}(C)/\sim} 1_D.$$

If necessary, e_C will be denoted by e_C^G. If H is a subgroup of G, then

$$\operatorname{Res}_H^G e_C^G = \sum_{D\in(C\cap H)/\sim_H} e_D^H. \tag{2.19}$$

Proposition 2.20. *Let $C \in G_{p'}/\!\sim$. Then $e_C \in \mathcal{O}_\mathfrak{p}\mathcal{R}(G)$.*

Proof. Using Brauer's Theorem, we only need to prove that $\mathrm{Res}_N^G e_C^G \in \mathcal{O}_\mathfrak{p}\mathcal{R}(N)$ for every nilpotent subgroup N of G. By 2.19, this amounts to prove the lemma whenever G is nilpotent. So we assume that G is nilpotent. Then $G = G_{p'} \times G_p$, and G_p and $G_{p'}$ are subgroups of G. Moreover, $C \subset G_{p'}$ and $\mathcal{S}_{p'}(G) = C \times G_p$. If we identify $K\mathcal{R}(G)$ and $K\mathcal{R}(G_{p'}) \otimes_K K\mathcal{R}(G_p)$, we have $e_C^G = 1_C^{G_{p'}} \otimes_{\mathcal{O}_\mathfrak{p}} e_1^{G_p}$. But, by 2.2, we have that $e_C^{G_{p'}} \in \mathcal{O}_\mathfrak{p}\mathcal{R}(G_{p'})$. On the other hand, $e_1^{G_p} = 1_{G_p} \in \mathcal{R}(G_p)$. The proof of the lemma is complete. □

Corollary 2.21. *Let $C \in G_{p'}/\!\sim$. Then e_C is a primitive idempotent of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$.*

Proof. By Proposition 2.15 (a), the number of primitive idempotents of $k\mathcal{R}(G)$ is $|G_{p'}/\!\sim|$. So the number of primitive idempotents of $\hat{\mathcal{O}}_\mathfrak{p}\mathcal{R}(G)$ is also $|G_{p'}/\!\sim|$ (here, $\hat{\mathcal{O}}_\mathfrak{p}$ denotes the completion of $\mathcal{O}_\mathfrak{p}$ at its maximal ideal). Now, $(e_C)_{C\in G_{p'}/\sim}$ is a family of orthogonal idempotents of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$ (see Proposition 2.20) and $1_G = \sum_{C\in G_{p'}/\sim} e_C$. The proof of the lemma is complete. □

Let $\bar{e}_C \in k\mathcal{R}(G)$ denote the reduction modulo $\mathfrak{p}\mathcal{O}_\mathfrak{p}$ of e_C. Then it follows from 2.12 that

(2.22) $$d_\mathfrak{p}^k \bar{e}_C = 1_{\mathcal{S}_{p'}(C)}^k \in k\mathcal{R}(kG) \simeq \mathrm{Class}_k^{p'}(G).$$

Let $\mathcal{P}_C = \mathcal{O}_\mathfrak{p}\mathcal{R}(G)e_C$ and $\bar{\mathcal{P}}_C = k\mathcal{R}(G)\bar{e}_C$: they are indecomposable projective modules for $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$ and $k\mathcal{R}(G)$ respectively. Then

$$\mathcal{O}_\mathfrak{p}\mathcal{R}(G) = \bigoplus_{C\in G_{p'}/\sim} \mathcal{P}_C$$

and

$$k\mathcal{R}(G) = \bigoplus_{C\in G_{p'}/\sim} \bar{\mathcal{P}}_C.$$

Note also that

(2.23) $$\dim_k k\mathcal{R}(G)\bar{e}_C = \mathrm{rank}_{\mathcal{O}_\mathfrak{p}} \mathcal{O}_\mathfrak{p}\mathcal{R}(G)e_C = |\mathcal{S}_{p'}(G)/\!\sim|.$$

Proposition 2.24. *Let C and C' be two conjugacy classes of p'-regular elements of G. Then:*

(a) $[\bar{\mathcal{P}}_C : \bar{\mathcal{D}}_{C'}] = \begin{cases} |\mathcal{S}_{p'}(C)/\!\sim| & \textit{if } C = C', \\ 0 & \textit{otherwise.} \end{cases}$

(b) $\bar{\mathcal{P}}_C/\operatorname{Rad}\bar{\mathcal{P}}_C \simeq \bar{\mathcal{D}}_C$.

Proof. Let us first prove (a). By definition of e_C, we have

$$[K \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{P}_C] = \sum_{D\in\mathcal{S}_{p'}(G)/\sim} [\mathcal{D}_D].$$

Also, by definition of the decomposition map $\delta_\mathfrak{p} : \mathcal{R}(K\mathcal{R}(G)) \to \mathcal{R}(k\mathcal{R}(G))$, we have

$$\delta_\mathfrak{p}[K \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{P}_C] = [\bar{\mathcal{P}}_C].$$

So the result follows from these observations and from 2.13. Now, (b) follows easily from (a). □

2.E. **More on the radical.** Let $\mathrm{Rad}_p(G)$ denote the set of functions $f \in \mathcal{O}_\mathfrak{p}\mathcal{R}(G)$ whose restriction to $G_{p'}$ is zero. Note that $\mathrm{Rad}_p(G)$ is a direct summand of the $\mathcal{O}_\mathfrak{p}$-module $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$. So, $k\,\mathrm{Rad}_p(G) = k \otimes_{\mathcal{O}_\mathfrak{p}} \mathrm{Rad}_p(G)$ is a sub-k-vector space of $k\mathcal{R}(G)$.

Proposition 2.25. *We have:*

(a) $\dim_k k\,\mathrm{Rad}_p(G) = |G/\sim| - |G_{p'}/\sim|$.
(b) $k\,\mathrm{Rad}_p(G)$ *is the radical of* $k\mathcal{R}(G)$.

Proof. (a) is clear. (b) follows from 2.12 and from Corollary 2.15. $\square$

Corollary 2.26. *Let e be the number such that p^e is the exponent of a Sylow p-subgroup of G. If $f \in \mathrm{Rad}\,k\mathcal{R}(G)$, then $f^{p^e} = 0$.*

Proof. Let $e = e_p(G)$. If $f \in K\mathcal{R}(G)$ and if $n \geqslant 1$, we denote by $f^{(n)} : G \to K$, $g \mapsto f(g^n)$. Then the map $K\mathcal{R}(G) \to K\mathcal{R}(G)$, $f \mapsto f^{(n)}$ is a morphism of K-algebras. Moreover (see for instance [CR, Corollary 12.10]), we have

(2.27) *If $f \in \mathcal{R}(G)$, then $f^{(n)} \in \mathcal{R}(G)$.*

Therefore, it induces a morphism of k-algebras $\theta_n : k\mathcal{R}(G) \to k\mathcal{R}(G)$. Now, let $F : k\mathcal{R}(G) \to k\mathcal{R}(G)$, $\lambda \otimes_\mathbb{Z} f \mapsto \lambda^p \otimes_\mathbb{Z} f$. Then F is an injective endomorphism of the ring $k\mathcal{R}(G)$. Moreover (see for instance [I, Problem 4.7]), we have

(2.28) $$F \circ \theta_p(f) = f^p$$

for every $f \in k\mathcal{R}(G)$. Since F and θ_p commute, we have $F^e \circ \theta_{p^e}(f) = f^{p^e}$ for every $f \in k\mathcal{R}(G)$. Therefore, if $\chi \in \mathrm{Rad}_p(G)$, we have

$$\bar{\chi}^{p^e} = F^e(\overline{\chi^{(p^e)}}).$$

But, by hypothesis, $g^{p^e} \in G_{p'}$ for every $g \in G$. So, if $f \in \mathrm{Rad}_p(G)$, then $f^{(p^e)} = 0$. Therefore, $\bar{f}^{p^e} = 0$. The corollary follows from this observation and from Proposition 2.25. $\square$

3. Principal block

If $C \in G_{p'}/\sim$, we denote by $\mathcal{R}_\mathfrak{p}(G,C)$ the $\mathcal{O}_\mathfrak{p}$-algebra $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)e_C$. As an $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$-module, this is just $\mathcal{P}_C$, but we want to study here its structure as a ring, so that is why we use a different notation. If R is a commutative $\mathcal{O}_\mathfrak{p}$-algebra, we set $R\mathcal{R}_\mathfrak{p}(G,C) = R \otimes_{\mathcal{O}_\mathfrak{p}} \mathcal{R}_\mathfrak{p}(G,C)$. For instance, $k\mathcal{R}_\mathfrak{p}(G,C) = k\mathcal{R}(G)\bar{e}_C$, and $K\mathcal{R}_\mathfrak{p}(G,C)$ can be identified with the algebra of class functions on $\mathcal{S}_{p'}(C)$.

The algebra $\mathcal{R}_\mathfrak{p}(G,1)$ (resp. $k\mathcal{R}_\mathfrak{p}(G,1)$) will be called the *principal block* of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$ (resp. $k\mathcal{R}(G)$). The aim of this section is to construct an isomorphism $\mathcal{R}_\mathfrak{p}(G,C) \simeq \mathcal{R}_\mathfrak{p}(C_G(g),1)$, where g is any element of C. We also emphasize the functorial properties of the principal block.

REMARK 3.1 - If $C \in G_{p'}/\sim$ and if $z \in Z(G)$, then t_z induces an isomorphism of algebras $\mathcal{R}_{\mathfrak{p}}(G,C) \simeq \mathcal{R}_{\mathfrak{p}}(G, z_{p'}^{-1}C)$ (see Remark 2.7). Consequently, $\bar{t}_z$ induces an isomorphism of algebras $k\mathcal{R}_{\mathfrak{p}}(G,C) \simeq k\mathcal{R}_{\mathfrak{p}}(G, z^{-1}C)$. □

3.A. **Centralizers.** Let $C \in G_{p'}/\sim_G$. Let $\mathrm{proj}_C^G : K\mathcal{R}(G) \to K\mathcal{R}_{\mathfrak{p}}(G,C)$, $x \mapsto xe_C$ denote the canonical projection. We still denote by $\mathrm{proj}_C^G : \mathcal{O}_{\mathfrak{p}}\mathcal{R}(G) \to \mathcal{R}_{\mathfrak{p}}(G,C)$, the restriction of proj_C^G and we denote by $\overline{\mathrm{proj}_C^G} : k\mathcal{R}(G) \to k\mathcal{R}_{\mathfrak{p}}(G,C)$ its reduction modulo $\mathfrak{p}\mathcal{O}_{\mathfrak{p}}$.

Let us now fix $g \in C$. It is well-known (and easy) that the map $C_G(g)_p/\sim_{C_G(g)} \to \mathcal{S}_{p'}(C)/\sim_G$ that sends the $C_G(g)$-conjugacy class $D \in C_G(g)_p/\sim_{C_G(g)}$ to the G-conjugacy class containing gD is bijective. In particular,

(3.2) $$|\mathcal{S}_{p'}(C)/\sim_G| = |C_G(g)_p/\sim_{C_G(g)}|.$$

Now, let $d_g^G : K\mathcal{R}(G) \to K\mathcal{R}(C_G(g))$ be the map defined by:

$$(d_g^G f)(h) = \begin{cases} f(gh) & \text{if } h \in C_G(g)_p, \\ 0 & \text{otherwise,} \end{cases}$$

for all $f \in K\mathcal{R}(G)$ and $h \in C_G(g)$. Then $d_g^G f \in K\mathcal{R}_{\mathfrak{p}}(C_G(g),1)$. It must be noticed that

(3.3) $$d_g^G = \mathrm{proj}_1^{C_G(g)} \circ t_g^{C_G(g)} \circ \mathrm{Res}_{C_G(g)}^G = t_g^{C_G(g)} \circ \mathrm{proj}_g^{C_G(g)} \circ \mathrm{Res}_{C_G(g)}^G.$$

In particular, d_g^G sends $\mathcal{O}_{\mathfrak{p}}\mathcal{R}(G)$ to $\mathcal{R}_{\mathfrak{p}}(C_G(g),1)$. We denote by $\mathrm{res}_g : \mathcal{R}_{\mathfrak{p}}(G,C) \to \mathcal{R}_{\mathfrak{p}}(C_G(g),1)$ the restriction of d_g^G to $\mathcal{R}_{\mathfrak{p}}(G,C)$. Let $\mathrm{ind}_g : K\mathcal{R}_{\mathfrak{p}}(C_G(g),1) \to K\mathcal{R}_{\mathfrak{p}}(G,C)$ be the map defined by

$$\mathrm{ind}_g f = \mathrm{Ind}_{C_G(g)}^G (t_{g^{-1}}^{C_G(g)} f)$$

for every $f \in K\mathcal{R}_{\mathfrak{p}}(C_G(g),1)$. It is clear that $\mathrm{ind}_g f \in \mathcal{R}_{\mathfrak{p}}(G,C)$ if $f \in \mathcal{R}_{\mathfrak{p}}(C_G(g),1)$. Thus we have defined two maps

$$\mathrm{res}_g : \mathcal{R}_{\mathfrak{p}}(G,C) \to \mathcal{R}_{\mathfrak{p}}(C_G(g),1)$$

and $$\mathrm{ind}_g : \mathcal{R}_{\mathfrak{p}}(C_G(g),1) \to \mathcal{R}_{\mathfrak{p}}(G,C).$$

We have:

Theorem 3.4. *If $g \in G_{p'}$, then res_g and ind_g are isomorphisms of $\mathcal{O}_{\mathfrak{p}}$-algebras inverse to each other.*

Proof. We first want to prove that $\mathrm{res}_g \circ \mathrm{ind}_g$ is the identity. Let $f \in K\mathcal{R}_{\mathfrak{p}}(C_G(g),1)$. Let $f' = t_{g^{-1}} f$ and let $x \in C_G(g)_p$. We just need to prove that

(?) $$(\mathrm{Ind}_{C_G(g)}^G f')(gx) = f'(gx).$$

But, by definition,

$$(\mathrm{Ind}_{C_G(g)}^G f')(gx) = \sum_{\substack{h \in [G/C_G(g)] \\ h(gx)h^{-1} \in C_G(g)}} f'(h(gx)h^{-1}).$$

Here, $[G/C_G(g)]$ denotes a set of representatives of $G/C_G(g)$. Since f' has support in $gC_G(g)_p$, we have $f(h(gx)h^{-1}) \neq 0$ only if the p'-part of $h(gx)h^{-1}$ is equal to g, which happens if and only if $h \in C_G(g)$. This shows (?).

The fact that $\mathrm{ind}_g \circ \mathrm{res}_g$ is the identity can be proved similarly, or can be proved by using a trivial dimension argument. Since res_g is a morphism of algebras, we get that ind_g is also a morphism of algebras. $\square$

3.B. **Subgroups of index prime to p.** If H is a subgroup of G, then the restriction map Res_H^G sends $\mathcal{R}_\mathfrak{p}(G,1)$ to $\mathcal{R}_\mathfrak{p}(H,1)$ (indeed, by 2.19, we have $\mathrm{Res}_H^G e_1^G = e_1^H$).

Theorem 3.5. *If H is a subgroup of G of index prime to p, then $\mathrm{Res}_H^G : \mathcal{R}_\mathfrak{p}(G,1) \to \mathcal{R}_\mathfrak{p}(H,1)$ is a split injection of $\mathcal{O}_\mathfrak{p}$-modules.*

Proof. Let us first prove that Res_H^G is injective. For this, we only need to prove that the map $\mathrm{Res}_H^G : K\mathcal{R}_\mathfrak{p}(G,1) \to K\mathcal{R}_\mathfrak{p}(H,1)$. But $K\mathcal{R}_\mathfrak{p}(G,1)$ is the space of functions whose support is contained in G_p. Since the index of H is prime to p, every conjugacy class of p-elements of G meets H. This shows that Res_H^G is injective.

In order to prove that it is a split injection, we only need to prove that the $\mathcal{O}_\mathfrak{p}$-module $\mathcal{R}_\mathfrak{p}(H,1)/\mathrm{Res}_H^G(\mathcal{R}_\mathfrak{p}(G,1))$ is torsion-free. Let π be a generator of the ideal $\mathfrak{p}\mathcal{O}_\mathfrak{p}$. Let $\gamma \in \mathcal{R}_\mathfrak{p}(G,1)$ and $\eta \in \mathcal{R}_\mathfrak{p}(H,1)$ be such that $\pi\eta = \mathrm{Res}_H^G \gamma$. We only need to prove that $\gamma/\pi \in \mathcal{R}_\mathfrak{p}(G,1)$. By Brauer's Theorem, it is sufficient to show that, for any nilpotent subgroup N of G, we have $\mathrm{Res}_N^G \gamma \in \pi\mathcal{O}_\mathfrak{p}\mathcal{R}(N)$.

So let N be a nilpotent subgroup. We have $N = N_p \times N_{p'}$ and, since the index of H in G is prime to p, we may assume that $N_p \subset H$. Since $\mathrm{Res}_N^G \psi \in \mathcal{R}_\mathfrak{p}(N,1) = \mathcal{O}_\mathfrak{p}\mathcal{R}(N_p) \otimes_{\mathcal{O}_\mathfrak{p}} e_1^{N_{p'}}$, we have

$$\begin{aligned}\mathrm{Res}_N^G \gamma &= (\mathrm{Res}_{N_p}^G \gamma) \otimes_{\mathcal{O}_\mathfrak{p}} e_1^{N_{p'}} \\ &= (\pi\, \mathrm{Res}_{N_p}^H \eta) \otimes_{\mathcal{O}_\mathfrak{p}} e_1^{N_{p'}} \in \pi\mathcal{O}_\mathfrak{p}\mathcal{R}(N),\end{aligned}$$

as expected. $\square$

Corollary 3.6. *If H is a subgroup of G of index prime to p, then the map $\overline{\mathrm{Res}}_H^G : k\mathcal{R}_\mathfrak{p}(G,1) \to k\mathcal{R}_\mathfrak{p}(H,1)$ is an injective morphism of k-algebras.*

Corollary 3.7. *If H is a subgroup of G of index prime to p which controls the fusion of p-elements, then $\mathrm{Res}_H^G : \mathcal{R}_\mathfrak{p}(G,1) \to \mathcal{R}_\mathfrak{p}(H,1)$ is an isomorphism of $\mathcal{O}_\mathfrak{p}$-algebras.*

Proof. In this case, $\dim_K K\mathcal{R}_\mathfrak{p}(G,1) = \dim_K K\mathcal{R}_\mathfrak{p}(H,1)$, so the result follows from Corollary 3.6. $\square$

EXAMPLE 3.8 - Let P be a Sylow p-subgroup of G and assume in this example that P is abelian. Then $N_G(P)$ controls the fusion of p-elements. It then follows from Corollary

3.7 that the restriction from G to $N_G(P)$ induces isomorphisms of algebras $\mathcal{R}_\mathfrak{p}(G,1) \simeq \mathcal{R}_\mathfrak{p}(N_G(P),1)$ and $k\mathcal{R}_\mathfrak{p}(G,1) \simeq k\mathcal{R}_\mathfrak{p}(N_G(P),1)$. In particular, $\ell_p(G,1) = \ell_p(N_G(P),1)$. □

EXAMPLE 3.9 - Let N be a p'-group, let H be a group acting on N and let $G = H \ltimes N$. Then H is of index prime to p and controls the fusion of p-elements of G. So Res_H^G induces isomorphisms of algebras $\mathcal{R}_\mathfrak{p}(G,1) \simeq \mathcal{R}_\mathfrak{p}(H,1)$ and $k\mathcal{R}_\mathfrak{p}(G,1) \simeq k\mathcal{R}_\mathfrak{p}(H,1)$. In particular, $\ell_p(G,1) = \ell_p(H,1)$. □

3.C. **Quotient by a normal p'-subgroup.** Let N be a normal subgroup of G. Let $\pi : G \to G/N$ denote the canonical morphism. Then the morphism of algebras $\mathrm{Res}_\pi : \mathcal{R}_\mathfrak{p}(G/N) \to \mathcal{R}_\mathfrak{p}(G)$ induces a morphism of algebras $\mathrm{Res}_\pi^{(1)} : \mathcal{R}_\mathfrak{p}(G/N,1) \to \mathcal{R}_\mathfrak{p}(G,1)$, $f \mapsto (\mathrm{Res}_\pi f)e_1^G$. Note that $\mathrm{Res}_\pi^{(1)} e_1^{G/N} = e_1^G$. We denote by $\overline{\mathrm{Res}}_\pi^{(1)} : k\mathcal{R}_\mathfrak{p}(G/N,1) \to k\mathcal{R}_\mathfrak{p}(G,1)$ the morphism induced by $\mathrm{Res}_\pi^{(1)}$. Then:

Theorem 3.10. *With the above notation, we have:*

(a) $\mathrm{Res}_\pi^{(1)}$ *is a split injection of $\mathcal{O}_\mathfrak{p}$-modules.*
(b) *If N is prime to p, then* $\mathrm{Res}_\pi^{(1)}$ *is an isomorphism.*

Proof. (a) The injectivity of $\mathrm{Res}_\pi^{(1)}$ follows from the fact that $(G/N)_p = G_pN/N$. Now, let I denote the image of $\mathrm{Res}_\pi^{(1)}$. Since $\mathrm{Res}_\pi(\mathcal{O}_\mathfrak{p}\mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$, we get that $\mathrm{Res}_\pi(\mathcal{R}_\mathfrak{p}(G/N,1)) = (\mathrm{Res}_\pi^{(1)} e_1^{G/N})\,\mathrm{Res}_\pi(\mathcal{O}_\mathfrak{p}\mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$. Since $I = e_1^G \,\mathrm{Res}_\pi(\mathcal{R}_\mathfrak{p}(G/N,1))$ and $e_1^G = e_1^G \,\mathrm{Res}_\pi(e_1^{G/N})$, we get that $I = e_1^G \,\mathrm{Res}_\pi(\mathcal{O}_\mathfrak{p}\mathcal{R}(G/N))$ is a direct summand of $\mathcal{O}_\mathfrak{p}\mathcal{R}(G)$, as desired.

(b) now follows from (a) and from the fact that the map π induces a bijection between $G_p/{\sim_G}$ and $(G/N)_g/{\sim_{G/N}}$ whenever N is a normal p'-subgroup. □

4. SOME INVARIANTS

We introduce in this section some numerical invariants of the k-algebra $k\mathcal{R}(G)$ (more precisely, of the algebras $k\mathcal{R}_\mathfrak{p}(G,C)$): Loewy length, dimension of the Ext-groups.

4.A. **Loewy length.** If $C \in G_{p'}/\sim$, we denote by $\ell_p(G,C)$ the Loewy length of the k-algebra $k\mathcal{R}_\mathfrak{p}(G,C)$. Then, by definition, we have

(4.1) $$\ell_p(G) = \max_{C \in G_{p'}/\sim} \ell_p(G,C).$$

On the other hand, by Theorem 3.4, we have

(4.2) *If $C \in G_{p'}/\sim$ and if $g \in C$, then $\ell_p(G,C) = \ell_p(C_G(g),1)$.*

The following bound on the Loewy length of $k\mathcal{R}(G)$ is obtained immediately from 2.23 and 3.2:

(4.3) $$\ell_p(G) \leqslant \max_{C \in G_{p'}/\sim} |\mathcal{S}_{p'}(C)/\sim| = \max_{g \in G_{p'}} |C_G(g)_p/\sim_{C_G(g)}|.$$

We set $S_p(G) = \max_{C \in G_{p'}/\sim} |\mathcal{S}_{p'}(C)/\sim|$.

EXAMPLE 4.4 - The inequality 4.3 might be strict. Indeed, if $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, then $\ell_2(G) = 3 < 4 = S_2(G)$. $\square$

EXAMPLE 4.5 - If $S_p(G) = 2$, then $\ell_p(G) = 2$. Indeed, in this case, we have that p divides $|G|$, so $k\mathcal{R}(G)$ is not semisimple by Corollary 2.17, so $\ell_p(G) \geqslant 2$. The result then follows from 4.3. $\square$

4.B. **Ext-groups.** If $i \geqslant 0$ and if $C \in G_{p'}/\sim$, we set

$$\mathrm{ext}_p^i(G,C) = \dim_{\mathbb{F}_p} \mathrm{Ext}^i_{k\mathcal{R}(G)}(\bar{\mathcal{D}}_C, \bar{\mathcal{D}}_C).$$

Note that $\mathrm{ext}_p^i(G,C) = \dim_{\mathbb{F}_p} \mathrm{Ext}^i_{k\mathcal{R}_\mathfrak{p}(G,C)}(\bar{\mathcal{D}}_C, \bar{\mathcal{D}}_C)$. So, if $g \in C$, it follows from Theorem 3.4 that

(4.6) $$\mathrm{ext}_p^i(G,C) = \mathrm{ext}_p^i(C_G(g), 1).$$

4.C. **Subgroups, quotients.** The next results follows respectively from Corollaries 3.6, 3.7 and from Theorem 3.10:

Proposition 4.7. *Let H be a subgroup of G of index prime to p and let N be a normal subgroup of G. Then:*

(a) $\ell_p(G,1) \leqslant \ell_p(H,1)$.
(b) *If H controls the fusion of p-elements, then $\ell_p(G,1) = \ell_p(H,1)$ and $\mathrm{ext}_p^i(G,1) = \mathrm{ext}_p^i(H,1)$ for every $i \geqslant 0$.*
(c) $\ell_p(G/N,1) \leqslant \ell_p(G,1)$.
(d) *If $|N|$ is prime to p, then $\ell_p(G,1) = \ell_p(H,1)$ and $\mathrm{ext}_p^i(G,1) = \mathrm{ext}_p^i(H,1)$ for every $i \geqslant 0$.*

4.D. **Direct products.** We study here the behaviour of the invariants $\ell_p(G,C)$ and $\mathrm{ext}_p^1(G,C)$ with respect to taking direct products. We first recall the following result on finite dimensional algebras:

Proposition 4.8. *Let A and B be two finite dimensional k-algebras. Then:*

(a) $\mathrm{Rad}(A\otimes_k B) = A\otimes_k (\mathrm{Rad}\, B) + (\mathrm{Rad}\, A)\otimes_k B$.

(b) *If $A/\mathrm{Rad}\, A \simeq k$ and $B/\mathrm{Rad}\, B\simeq k$, then*

$$\mathrm{Rad}(A\otimes_k B)/\mathrm{Rad}(A\otimes_k B)^2 \simeq (\mathrm{Rad}\, A)/(\mathrm{Rad}\, A)^2 \oplus (\mathrm{Rad}\, B)/(\mathrm{Rad}\, B)^2.$$

Proof. (a) is proved for instance in [CR, Proof of 10.39]. Let us now prove (b). Let $\theta : (\mathrm{Rad}\, A)\oplus(\mathrm{Rad}\, B) \to \mathrm{Rad}(A\otimes_k B)/\mathrm{Rad}(A\otimes_k B)^2$, $a\oplus b\mapsto \overline{a\otimes_k 1 + 1\otimes_k b}$. By (a), θ is surjective and $(\mathrm{Rad}\, A)^2\oplus(\mathrm{Rad}\, B)^2$ is contained in the kernel of θ. Now the result follows from dimension reasons (using (a)). $\square$

Proposition 4.9. *Let G and H be two finite groups and let $C\in G_{p'}/\!\sim$ and $D\in H_{p'}/\!\sim$. Then*

$$\ell_p(G\times H, C\times D) = \ell_p(G,C) + \ell_p(H,D) - 1$$

and

$$\mathrm{ext}_p^1(G\times H, C\times D) = \mathrm{ext}_p^1(G,C) + \mathrm{ext}_p^1(H,D).$$

Proof. Write $A = k\mathcal{R}_\mathfrak{p}(G,C)$ and $B = k\mathcal{R}_\mathfrak{p}(H,D)$. It is easily checked that $k\mathcal{R}_\mathfrak{p}(G\times H, C\times D) = A\otimes_k B$. So the first equality follows from Propositon 4.8 (a) and from the commutativity of A and B. Moreover $A/(\mathrm{Rad}\, A)\simeq k$ and $B/(\mathrm{Rad}\, B)\simeq k$. In particular $\dim_k \mathrm{Ext}_A^1(A/\mathrm{Rad}\, A, A/\mathrm{Rad}\, A) = \dim_k(\mathrm{Rad}\, A)/(\mathrm{Rad}\, A)^2$. So the second equality follows from Proposition 4.8 (b). $\square$

4.E. **Abelian groups.** We compute here the invariants $\ell_p(G,1)$ and $\mathrm{ext}_p^1(G,1)$ whenever G is abelian. If G is abelian, then there is a (non-canonical) isomorphism of algebras $k\mathcal{R}(G)\simeq kG$.

Let us first start with the cyclic case:

(4.10) *if G is cyclic, then $\ell_p(G) = |G|_p + 1$ and* $\mathrm{ext}_p^1(G,1) = \begin{cases} 1 & \textit{if } p \textit{ divides } |G|,\\ 0 & \textit{otherwise.}\end{cases}$

Therefore, by Proposition 4.9, we have: if G_1, …, G_n are cyclic, then

(4.11) $$\ell_p(G_1\times\cdots\times G_n) = |G_1|_p + \cdots + |G_n|_p - n + 1.$$

and

(4.12) $$\mathrm{ext}_p^1(G_1\times\cdots\times G_n) = |\{1\leqslant i\leqslant n \mid p \textit{ divides } G_i\}|.$$

5. The symmetric group

In this section, and only in this section, we fix a non-zero natural number n and a prime number p and we assume that $G = \mathfrak{S}_n$, that $\mathcal{O} = \mathbb{Z}$ and that $\mathfrak{p} = p\mathbb{Z}$. Let $\mathbb{F}_p = k$. It is well-known that $\mathbb{Q}$ and $\mathbb{F}_p$ are splitting fields for $\mathfrak{S}_n$. For simplification, we set $\mathcal{R}_n = \mathcal{R}(\mathfrak{S}_n)$ and $\overline{\mathcal{R}}_n = \mathbb{F}_p\mathcal{R}(\mathfrak{S}_n)$. We investigate further the structure of $\overline{\mathcal{R}}_n$. This is a continuation of the work started in [B] in which the description of the descending Loewy series of $\overline{\mathcal{R}}_n$ was obtained.

We first introduce some notation. Let $\mathrm{Part}(n)$ denote the set of partitions of n. If $\lambda = (\lambda_1, \dots, \lambda_r) \in \mathrm{Part}(n)$ and if $1 \leqslant i \leqslant n$, we denote by $r_i(\lambda)$ the number of occurences of i as a part of λ. We set

$$\pi_p(\lambda) = \sum_{i=1}^{n} \Big[\frac{r_i(\lambda)}{p}\Big]$$

where, for $x \in \mathbb{R}$, $x \geqslant 0$, we denote by $[x]$ the unique natural number $m \geqslant 0$ such that $m \leqslant x < m+1$. Note that $\pi_p(\lambda) \in \{0, 1, 2, \dots, [n/p]\}$ and recall that λ is *p-regular* (resp. *p-singular*) if and only if $\pi_p(\lambda) = 0$ (resp. $\pi_p(\lambda) \geqslant 1$). We denote by $\mathfrak{S}_\lambda$ the Young subgroup canonically isomorphic to $\mathfrak{S}_{\lambda_1} \times \cdots \times \mathfrak{S}_{\lambda_r}$, by 1_λ the trivial character of $\mathfrak{S}_\lambda$, and by c_λ an element of $\mathfrak{S}_\lambda$ with only r orbit in $\{1, 2, \dots, n\}$. Let C_λ denote the conjugacy class of c_λ in $\mathfrak{S}_n$. Then the map $\mathrm{Part}(n) \to \mathfrak{S}_n/{\sim}$, $\lambda \mapsto C_\lambda$ is a bijection. Let $W(\lambda) = N_{\mathfrak{S}_n}(\mathfrak{S}_\lambda)/\mathfrak{S}_\lambda$. Then

(5.1)
$$W(\lambda) \simeq \prod_{i=1}^{n} \mathfrak{S}_{r_i(\lambda)}.$$

In particular, $\pi_p(\lambda)$ is the *p-rank* of $W(\lambda)$, where the p-rank of a finite group is the maximal rank of an elementary abelian subgroup. Now, we set $\varphi_\lambda = \mathrm{Ind}_{\mathfrak{S}_\lambda}^{\mathfrak{S}_n} 1_\lambda$. An old result of Frobenius says that

(5.2) $(\varphi_\lambda)_{\lambda \in \mathrm{Part}(n)}$ *is a $\mathbb{Z}$-basis of $\mathcal{R}_n$*

(see for instance [GP, Theorem 5.4.5 (b)]). Now, if $i \geqslant 1$, let

$$\mathrm{Part}_p^{\geqslant i}(n) = \{\lambda \in \mathrm{Part}(n) \mid \pi_p(\lambda) \geqslant i\}$$

and
$$\mathrm{Part}_p^{i}(n) = \{\lambda \in \mathrm{Part}(n) \mid \pi_p(\lambda) = i\}.$$

Then, by [B, Theorem A], we have

(5.3)
$$(\mathrm{Rad}\,\overline{\mathcal{R}}_n)^i = \bigoplus_{\lambda \in \mathrm{Part}_p^{(i)}(n)} \mathbb{F}_p\bar{\varphi}_\lambda.$$

Let $\mathrm{Part}_{p'}(n)$ denote the set of partitions of n whose parts are prime to p. Then the map $\mathrm{Part}_{p'}(n) \to G_{p'}/{\sim}$, $\lambda \mapsto C_\lambda$ is bijective. We denote by $\tau_{p'}(\lambda)$ the unique partition of n such that $(c_\lambda)_{p'} \in C_{\tau_{p'}(\lambda)}$. If $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$, the partition $\tau_{p'}(\lambda)$ is obtained as follows. Let

$$\lambda' = (\underbrace{(\lambda_1)_{p'}, \dots, (\lambda_1)_{p'}}_{(\lambda_1)_p \text{ times}}, \dots, \underbrace{(\lambda_r)_{p'}, \dots, (\lambda_r)_{p'}}_{(\lambda_r)_p \text{ times}}).$$

Then $\tau_{p'}(\lambda)$ is obtained from λ' by reordering the parts. The map $\tau_{p'} : \mathrm{Part}(n) \to \mathrm{Part}_{p'}(n)$ is obviously surjective. If $\lambda \in \mathrm{Part}_{p'}(n)$, we set for simplification $\mathcal{R}_{n,p}(\lambda) = \mathcal{R}_{p\mathbb{Z}}(\mathfrak{S}_n, C_\lambda)$ and $\overline{\mathcal{R}}_n(\lambda) = \mathbb{F}_p\mathcal{R}_{p\mathbb{Z}}(\mathfrak{S}_n, C_\lambda)$. In other words,

$$\mathbb{Z}_{p\mathbb{Z}}\mathcal{R}_n = \bigoplus_{\lambda \in \mathrm{Part}_{p'}(n)} \mathcal{R}_{n,p}(\lambda)$$

and
$$\overline{\mathcal{R}}_n = \bigoplus_{\lambda \in \mathrm{Part}_{p'}(n)} \overline{\mathcal{R}}_n(\lambda)$$

are the decomposition of $\mathbb{Z}_{p\mathbb{Z}}\mathcal{R}_n$ and $\overline{\mathcal{R}}_n$ as a sum of blocks. We now make the result 5.3 more precise:

Proposition 5.4. *If $\lambda \in \mathrm{Part}_{p'}(n)$ and if $i \geqslant 0$, then*

$$\dim_{\mathbb{F}_p} \big(\mathrm{Rad}\, \overline{\mathcal{R}}_n(\lambda)\big)^i = |\tau_{p'}^{-1}(\lambda) \cap \mathrm{Part}_p^{\geqslant i}(n)|.$$

Proof. If λ and μ are two partitions of n, we write $\lambda \subset \mu$ if $\mathfrak{S}_\lambda$ is conjugate to a subgroup of $\mathfrak{S}_\mu$. This defines an order on $\mathrm{Part}(n)$. On the other hand, if $d \in \mathfrak{S}_n$, we denote by $\lambda \cap {}^d\mu$ the unique partition ν of n such that $\mathfrak{S}_\lambda \cap {}^d\mathfrak{S}_\mu$ is conjugate to $\mathfrak{S}_\nu$. Then, by the Mackey formula for tensor product (see for instance [CR, Theorem 10.18]), we have

(1) $$\varphi_\lambda \varphi_\mu = \sum_{d \in [\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu]} \varphi_{\lambda \cap {}^d\mu}.$$

Here, $[\mathfrak{S}_\lambda \backslash \mathfrak{S}_n / \mathfrak{S}_\mu]$ denotes a set of representatives of the $(\mathfrak{S}_\lambda, \mathfrak{S}_\mu)$-double cosets in $\mathfrak{S}_n$. This shows that, if we fixe $\lambda_0 \in \mathrm{Part}(n)$, then $\oplus_{\lambda \subset \lambda_0} \mathbb{Z}\varphi_\lambda$ and $\oplus_{\lambda \subsetneq \lambda_0} \mathbb{Z}\varphi_\lambda$ are sub-$\mathcal{R}(G)$-module of $\mathcal{R}(G)$. We denote by $\mathcal{D}_\lambda^\mathbb{Z}$ the quotient of these two modules. Then

(2) $$K \otimes_\mathbb{Z} \mathcal{D}_\lambda^\mathbb{Z} \simeq \mathcal{D}_{C_\lambda}.$$

This follows for instance from [GP, Proposition 2.4.4]. Consequently,

(3) $$k \otimes_\mathbb{Z} \mathcal{D}_\lambda^\mathbb{Z} \simeq \bar{\mathcal{D}}_{C_\lambda}.$$

It then follows from Proposition 2.14 that

(4) $$k \otimes_\mathbb{Z} \mathcal{D}_\lambda^\mathbb{Z} \simeq k \otimes_\mathbb{Z} \mathcal{D}_\mu^\mathbb{Z} \quad \textit{if and only if} \quad \tau_{p'}(\lambda) = \tau_{p'}(\mu).$$

Now the Theorem follows from easily from (3), (4) and 5.3. □

Now, if $\lambda \in \mathrm{Part}_{p'}(n)$, then $C_{\mathfrak{S}_n}(w_\lambda)$ contains a normal p'-subgroup N_λ such that $C_{\mathfrak{S}_n}(w_\lambda)/N_\lambda \simeq W(\lambda)$. We denote by 1^n the partition $(1, 1, \dots, 1)$ of n. It follows from Theorem 3.4 and Theorem 3.10 that

(5.5) $$\mathcal{R}_{n,p}(\lambda) \simeq \mathcal{R}_{p\mathbb{Z}}(W(\lambda), 1) \simeq \bigotimes_{i=1}^{n} \mathcal{R}_{r_i(\lambda),p}(1^{r_i(\lambda)})$$

and

$$\overline{\mathcal{R}}_n(\lambda) \simeq \overline{\mathcal{R}}(W(\lambda),1) \simeq \bigotimes_{i=1}^{n} \overline{\mathcal{R}}_{r_i(\lambda)}(1^{r_i(\lambda)}). \tag{5.6}$$

We denote by $\operatorname{Log}_p n$ the real number x such that $p^x = n$. Then:

Corollary 5.7. *If $\lambda \in \operatorname{Part}_{p'}(n)$, then*

$$\operatorname{ext}_p^1(\mathfrak{S}_n, C_\lambda) = \sum_{i=1}^{n} [\operatorname{Log}_p r_i(\lambda)]$$

and

$$\ell_p(\mathfrak{S}_n, C_\lambda) = \pi_p(\lambda) + 1.$$

Proof. By 5.6 and by Proposition 4.9, both equalities need only to be proved whenever $\lambda = (1^n)$. So we assume that $\lambda = (1^n)$.

Let us show the first equality. By Proposition 5.4, we are reduced to show that $|\tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^1(n)| = [\operatorname{Log}_p n]$. Let $r = [\operatorname{Log}_p n]$. In other words, we have $p^r \leqslant n < p^{r+1}$. If $1 \leqslant i \leqslant r$, write $n - p^i = \sum_{j=0}^{r} a_{ij} p^j$ with $0 \leqslant a_{ij} < p - 1$ (the a_{ij}'s are uniquely determined). Let

$$\lambda(i) = (\underbrace{p^r, \dots, p^r}_{a_{ir}\ \text{times}}, \dots, \underbrace{p^i, \dots, p^i}_{a_{ir}\ \text{times}} \underbrace{p^{i-1}, \dots, p^{i-1}}_{(p+a_{i-1,r})\ \text{times}}, \underbrace{p^{i-2}, \dots, p^{i-2}}_{a_{i-2,r}\ \text{times}}, \dots, \underbrace{1, \dots, 1}_{a_{0r}\ \text{times}}).$$

The result will follow from the following equality

$$\tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^1(n) = \{\lambda(1), \lambda(2), \dots, \lambda(r)\}. \tag{$*$}$$

So let us now prove $(*)$. Let $I = \{\lambda(1), \lambda(2), \dots, \lambda(r)\}$. It is clear that $I \subset \tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^1(n)$. Now, let $\lambda \in \tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^1(n)$. Then there exists a unique $i \in \{1, 2, \dots, r\}$ such that $r_{p^{i-1}}(\lambda) \geqslant p$. Moreover, $r_{p^{i-1}}(\lambda) < 2p$. So, if we set $r'_{p^j} = r_{r_j}(\lambda)$ if $j \neq i-1$ and $r'_{p^{i-1}} = r_{p^{i-1}}(\lambda) - p$, we get that $0 \leqslant r'_{p^j} \leqslant p - 1$ and $n - p^i = \sum_{j=0}^{r} r'_{p^j} p^j$. This shows that $r'_{p^j} = a_{ij}$, so $\lambda = \lambda(i)$.

Let us now show the second equality fo the Corollary. By Proposition 5.4, we only need to show that $|\tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^{[n/p]}(n)| \geqslant 1$. But in fact, it is clear that $\tau_{p'}^{-1}(1^n) \cap \operatorname{Part}_p^{[n/p]}(n) = \{1^n\}$. $\square$

Corollary 5.8. *We have*

$$\dim_{\mathbb{F}_p} \big(\operatorname{Rad} \overline{\mathcal{R}}_n(1^n)\big)^{[n/p]} = 1$$

and

$$\dim_{\mathbb{F}_p} \operatorname{Ext}^1_{\overline{\mathcal{R}}_n}(\bar{\mathcal{D}}_{1^n}, \bar{\mathcal{D}}_{1^n}) = [\operatorname{Log}_p n].$$

In particular, $\ell_p(\mathfrak{S}_n, 1) = \ell_p(\mathfrak{S}_n) = [n/p]$.

Proof. This is just a particular case of the previous corollary. The first equality has been obtained in the course of the proof of the previous corollary. $\square$

6. Dihedral groups

Let $n \geqslant 1$ and $m \geqslant 0$ be two natural numbers. We assume in this section, and only in this subsection, that $G = D_{2^n(2m+1)}$ is the dihedral group of order $2^n(2m+1)$ and that $p = 2$.

Proposition 6.1. *If $n \geqslant 1$ and $m \geqslant 0$ are natural numbers, then*

$$\ell_2(D_{2^n(2m+1)}, 1) = \begin{cases} 2 & \textit{if } n = 1, \\ 3 & \textit{if } n = 2, \\ 2^{n-2}+1 & \textit{if } n \geqslant 3. \end{cases}$$

and

$$\operatorname{ext}_2^1(D_{2^n(2m+1)}, 1) = \begin{cases} 1 & \textit{if } n = 1, \\ 2 & \textit{if } n = 2, \\ 3 & \textit{if } n \geqslant 3. \end{cases}$$

Proof. Let N be the normal subgroup of G of order $2m+1$. Then $G \simeq D_{2^n} \ltimes N$. So, by Proposition 4.7 (d), we may, and we will, assume that $m = 0$. If $n = 1$ or 2 the the result is easily checked. Therefore, we may, and we will, assume that $n \geqslant 3$.

Write $h = 2^{n-1}$. We have

$$G = < s, t \mid s^2 = t^2 = (st)^h = 1 > .$$

Let $H = < st >$ and $S = < s >$. Then $|H| = 2^{n-1} = h$ and $G = S \ltimes H$. We fix a primitive h-th root of unity $\zeta \in \mathcal{O}^\times$. If $i \in \mathbb{Z}$, we denote by ξ_i the unique linear character of H such that $\xi_i(st) = \zeta^i$. Then $\operatorname{Irr} H = \{\xi_0, \xi_1, \ldots, \xi_{h-1}\}$, and $\xi_0 = 1_H$.

Since $n \geqslant 3$, h is even and, if we write $h = 2h'$, then $h' = 2^{n-2}$ is also even. For $i \in \mathbb{Z}$, we set

$$\chi_i = \operatorname{Ind}_H^G \xi_i.$$

It is readily seen that $\chi_i = \chi_{-i}$, that $\chi_{i+h} = \chi_i$ and that

(6.2) $$\chi_i \chi_j = \chi_{i+j} + \chi_{i-j}.$$

Let ε (resp. ε_s, resp. ε_t) be the unique linear character of order 2 such that $\varepsilon(st) = 1$ (resp. $\varepsilon_s(s) = 1$, resp. $\varepsilon_t(t) = 1$). Then

$$\chi_0 = 1_G + \varepsilon,$$

$$\chi_{h'} = \varepsilon_s + \varepsilon_t,$$

and, if h' does not divide i,

$$\chi_i \in \operatorname{Irr} G.$$

Moreover, $|\operatorname{Irr} G| = h' + 3$ and

$$\operatorname{Irr} G = \{1_G, \varepsilon, \varepsilon_s, \varepsilon_t, \chi_1, \chi_2, \ldots, \chi_{h'-1}\}.$$

Finally, note that

(6.3) $$\varepsilon_s \chi_i = \varepsilon_t \chi_i = \chi_{i+h'}.$$

Let us start by finding a lower bound for $\ell_2(G)$. First, notice that the following equality holds: for all i, $j \in \mathbb{Z}$ and every $r \geqslant 0$, we have

(6.4) $$(\bar{\chi}_i + \bar{\chi}_j)^{2^r} = \bar{\chi}_{2^r i} + \bar{\chi}_{2^r j}.$$

> *Proof of 6.4.* Recall that $\bar{\chi}_i$ denotes the image of χ_i in $k\mathcal{R}(G)$. We proceed by induction on r. The case $r = 0$ is trivial. The induction step is an immediate consequence of 6.2. $\square$

Note also the following fact (which follows from Example 2.18):

(6.5) *If $i \in \mathbb{Z}$, then $\bar{\chi}_i \in \operatorname{Rad} k\mathcal{R}(G)$.*

Therefore,

(6.6) $$\ell_2(G) \geqslant 2^{n-2} + 1.$$

> *Proof of 6.6.* By 6.4, we have immediately that $(\bar{\chi}_0 + \bar{\chi}_1)^{2^{n-2}} = \bar{\chi}_0 + \bar{\chi}_{h'} \neq 0$ and, by 6.5, $\bar{\chi}_0 + \bar{\chi}_1 \in \operatorname{Rad} k\mathcal{R}(G)$. $\square$

By Example 2.18, we have

(6.7) *$(\bar{1}_G + \bar{\varepsilon}_s, \bar{\chi}_0, \bar{\chi}_1, \dots, \bar{\chi}_{h'})$ is a k-basis of $\operatorname{Rad} k\mathcal{R}(G)$.*

By 6.3 and 6.2, we get that

(6.8) *$(\bar{\chi}_i + \bar{\chi}_{i+2})_{0 \leqslant i \leqslant h'-2}$ is a k-basis of $(\operatorname{Rad} k\mathcal{R}(G))^2$.*

This shows that $\operatorname{ext}^1_p(G) = 3$, as expected. It follows that, if $n \geqslant 3$ and $2 \leqslant i \leqslant 2^{n-2} + 1$, then

(6.9) $$\dim_k \bigl(\operatorname{Rad} k\mathcal{R}(D_{2^n})\bigr)^i = 2^{n-2} + 1 - i$$

> *Proof of 6.9.* Let $d_i = \dim_k \bigl(\operatorname{Rad} k\mathcal{R}(D_{2^n})\bigr)^i$. By 6.8, we have $d_2 = 2^{n-2} - 1$. By 6.6, we have $d_{2^{n-2}} \geqslant 1$. Moreover, $d_1 > d_2 > d_3 > \dots$ So the proof of 6.9 is complete. $\square$

In particular, we get:

(6.10) *If $n \geqslant 3$, then $\bigl(\operatorname{Rad} k\mathcal{R}(D_{2^n})\bigr)^{2^{n-2}} = k(\bar{1}_{D_{2^n}} + \bar{\varepsilon} + \bar{\varepsilon}_s + \bar{\varepsilon}_t)$.*

and $\ell_2(D_{2^n}) = 2^{n-2} + 1$, as expected. $\square$

7. EXAMPLES

For $0 \leqslant i \leqslant \ell_p(G) - 1$, we set $d_i = \dim_k(\operatorname{Rad} k\mathcal{R}(G))^i$. Note that $d_0 = |G/\sim|$ and $d_0 - d_1 = |G_{p'}/\sim|$. In this section, we give tables containing the values $\ell_p(G)$, $\ell_p(G,1)$, $S_p(G)$, $\mathrm{ext}^1_p(G,1)$ and the sequence $(d_0, d_1, d_2, \dots)$ for various groups. These computations have been made using `GAP3` [GAP3].

These computations show that, if G satisfies at least one of the following conditions:

(1) $|G| \leqslant 200$;

(2) G is a subgroup of $\mathfrak{S}_8$;

(3) G is one of the groups contained in the next tables;

then $\ell_p(G,1) = \ell_p(N_G(P),1)$ (here, P denotes a Sylow p-subgroup of G). Note also that this equality holds if P is abelian (see Example 3.8).

Question. *Is it true that* $\ell_p(G,1) = \ell_p(N_G(P),1)$*?*

The first table contains the datas for the the exceptional Weyl groups, the second table is for the alternating groups $\mathfrak{A}_n$ for $5 \leqslant n \leqslant 12$, the third table is for some small finite simple groups, and the last table is for the groups $PSL(2,q)$ for q a prime power $\leqslant 27$.

G	$\|G\|$	p	$\ell_p(G)$	$S_p(G)$	$d_0, d_1, d_2, \dots$	$\ell_p(G,1)$	$\mathrm{ext}^1_p(G,1)$
$W(E_6)$	51840	2	5	10	$25, 19, 9, 3, 1$	5	3
	$2^7.3^4.5$	3	4	5	$25, 13, 4, 1$	4	2
		5	2	2	$25, 2$	2	1
$W(E_7)$	2903040	2	7	24	$60, 52, 35, 18, 7, 3, 1$	7	4
	$2^{10}.3^4.5.7$	3	4	5	$60, 30, 8, 2$	4	2
		5	2	2	$60, 6$	2	1
		7	2	2	$60, 2$	2	1
$W(E_8)$	696729600	2	8	32	$112, 100, 68, 36, 17, 7, 3, 1$	8	5
	$2^{14}.3^5.5^2.7$	3	5	8	$112, 65, 24, 7, 2$	5	2
		5	3	3	$112, 17, 2$	3	1
		7	2	2	$112, 4$	2	1
$W(F_4)$	1152	2	5	14	$25, 21, 12, 4, 1$	5	4
	$2^7.3^2$	3	3	4	$25, 11, 2$	3	2
$W(H_3)$	120	2	3	4	$10, 6, 1$	3	2
	$2^3.3.5$	3	2	2	$10, 2$	2	1
		5	3	3	$10, 4, 2$	3	1
$W(H_4)$	14400	2	4	7	$34, 24, 9, 1$	4	3
	$2^6.3^2.5^2$	3	3	3	$34, 11, 2$	3	1
		5	5	6	$34, 20, 11, 4, 2$	5	2

G	$\vert G\vert$	p	$\ell_p(G)$	$S_p(G)$	$d_0, d_1, d_2, \ldots$	$\ell_p(G,1)$	$\mathrm{ext}^1_p(G,1)$
$\mathfrak{A}_5$	60	2	2	2	$5, 1$	2	1
	$2^2.3.5$	3	2	2	$5, 1$	2	1
		5	3	3	$5, 2, 1$	3	1
$\mathfrak{A}_6$	360	2	3	3	$7, 2, 1$	3	1
	$2^3.3^2.5$	3	3	3	$7, 2, 1$	3	1
		5	3	3	$7, 2, 1$	3	1
$\mathfrak{A}_7$	2520	2	3	3	$9, 3, 1$	3	1
	$2^3.3^2.5.7$	3	3	3	$9, 3, 1$	3	1
		5	2	2	$9, 1$	2	1
		7	3	3	$9, 2, 1$	3	1
$\mathfrak{A}_8$	20160	2	4	5	$14, 6, 2, 1$	4	2
	$2^6.3^2.5.7$	3	3	3	$14, 6, 2$	3	1
		5	3	3	$14, 3, 1$	2	1
		7	3	3	$14, 2, 1$	3	1
$\mathfrak{A}_9$	181440	2	4	5	$18, 8, 3, 1$	4	2
	$2^6.3^4.5.7$	3	4	6	$18, 10, 3, 1$	4	3
		5	3	3	$18, 4, 1$	2	1
		7	2	2	$18, 1$	2	1
$\mathfrak{A}_{10}$	1814400	2	5	7	$24, 12, 6, 2, 1$	5	2
	$2^7.3^4.5^2.7$	3	4	6	$24, 13, 4, 1$	4	3
		5	3	3	$24, 4, 1$	3	1
		7	3	3	$24, 3, 1$	2	1
$\mathfrak{A}_{11}$	19958400	2	5	7	$31, 17, 8, 3, 1$	5	2
	$2^7.3^4.5^2.7.11$	3	4	5	$31, 16, 6, 1$	4	2
		5	3	3	$31, 6, 1$	3	1
		7	3	3	$31, 4, 1$	2	1
		11	3	3	$31, 2, 1$	3	1
$\mathfrak{A}_{12}$	239500800	2	6	10	$43, 25, 13, 6, 2, 1$	6	2
	$2^9.3^5.5^2.7.11$	3	5	8	$43, 22, 9, 2, 1$	5	3
		5	3	3	$43, 10, 2$	3	1
		7	3	3	$43, 5, 1$	2	1
		11	3	3	$43, 2, 1$	3	1

G	$\|G\|$	p	$\ell_p(G)$	$S_p(G)$	$d_0, d_1, d_2, \ldots$	$\ell_p(G,1)$	$\mathrm{ext}^1_p(G,1)$
$GL(3,2)$	168	2	3	3	$6, 2, 1$	3	1
	$2^3.3.7$	3	2	2	$6, 1$	2	1
		7	3	3	$6, 2, 1$	3	1
$SL(2,8)$	504	2	2	2	$9, 1$	2	1
	$2^3.3^2.7$	3	5	5	$9, 4, 3, 2, 1$	5	1
		7	4	4	$9, 3, 2, 1$	4	1
$SL(3,3)$	5616	2	5	5	$12, 5, 3, 2, 1$	5	1
	$2^4.3^3.13$	3	3	3	$12, 3, 1$	3	1
		13	5	5	$12, 4, 3, 2, 1$	5	1
$SU(3,3)$	6048	2	6	7	$14, 9, 6, 4, 2, 1$	6	2
	$2^5.3^3.7$	3	3	3	$14, 5, 1$	3	1
		7	3	3	$14, 2, 1$	3	1
M_{11}	7920	2	5	5	$10, 5, 3, 2, 1$	5	1
	$2^4.3^2.5.11$	3	2	2	$10, 2$	2	1
		5	2	2	$10, 1$	2	1
		11	3	3	$10, 2, 1$	3	1
$PSp(4,3)$	25920	2	4	5	$20, 12, 5, 1$	4	2
	$2^6.3^4.5$	3	5	7	$20, 14, 8, 3, 1$	5	2
		5	2	2	$20, 1$	2	1
M_{12}	95040	2	4	7	$15, 9, 3, 1$	4	3
	$2^6.3^3.5.11$	3	3	3	$15, 4, 1$	3	1
		5	2	2	$15, 2$	2	1
		11	3	3	$15, 2, 1$	3	1
J_1	175560	2	2	2	$15, 4$	2	1
	$2^3.3.5.7.11.19$	3	2	2	$15, 4$	2	1
		5	3	3	$15, 6, 3$	3	1
		7	2	2	$15, 1$	2	1
		11	2	2	$15, 1$	2	1
		19	4	4	$15, 3, 2, 1$	4	1
M_{22}	443520	2	4	5	$12, 5, 2, 1$	4	2
	$2^7.3^2.5.7.11$	3	2	2	$12, 2$	2	1
		5	2	2	$12, 1$	2	1
		7	3	3	$12, 2, 1$	3	1
		11	3	3	$12, 2, 1$	3	1
J_2	604800	2	4	5	$21, 11, 3, 1$	4	2
	$2^7.3^3.5^2.7$	3	3	3	$21, 7, 1,$	3	1
		5	5	5	$21, 10, 6, 2, 1$	5	1
		7	2	2	$21, 1$	2	1
HS	44352000	2	5	9	$24, 15, 8, 3, 1$	5	3
	$2^9.3^2.5^3.7.11$	3	2	2	$24, 5$	2	1
		5	3	4	$24, 8, 2$	3	2
		7	2	2	$24, 1$	2	1
		11	3	3	$24, 2, 1$	3	1

G	$\|G\|$	p	$\ell_p(G)$	$S_p(G)$	$d_0, d_1, d_2, \ldots$	$\ell_p(G,1)$	$\mathrm{ext}_p^1(G,1)$
$PSL(2,2)$	6	2	2	2	$3,1$	2	1
$\simeq \mathfrak{S}_3$	2.3	3	2	2	$3,1$	2	1
$PSL(2,3)$	12	2	2	2	$4,1$	2	1
$\simeq \mathfrak{A}_4$	$2^2.3$	3	3	3	$4,2,1$	3	1
$PSL(2,4)$	60	2	2	2	$5,1$	2	1
$\simeq PSL(2,5)$	$2^2.3.5$	3	2	2	$5,1$	2	1
$\simeq \mathfrak{A}_5$		5	3	3	$5,2,1$	3	1
$PSL(2,7)$	168	2	3	3	$6,2,1$	3	1
	$2^3.3.7$	3	2	2	$6,1$	2	1
		7	3	3	$6,2,1$	3	1
$PSL(2,8)$	504	2	2	2	$9,1$	2	1
	$2^3.3^2.7$	3	5	5	$9,4,3,2,1$	5	1
		7	4	4	$9,3,2,1$	4	1
$PSL(2,9)$	360	2	3	3	$7,2,1$	3	1
$\simeq \mathfrak{A}_6$	$2^3.3^2.5$	3	3	3	$7,2,1$	3	1
		5	3	3	$7,2,1$	3	1
$PSL(2,11)$	660	2	2	2	$8,2$	2	1
	$2^2.3.5.11$	3	2	2	$8,2$	2	1
		5	3	3	$8,2,1$	3	1
		11	3	3	$8,2,1$	3	1
$PSL(2,13)$	1092	2	2	2	$9,2$	2	1
	$2^2.3.7.13$	3	2	2	$9,2$	2	1
		7	4	4	$9,3,2,1$	4	1
		13	3	3	$9,2,1$	3	1
$PSL(2,16)$	4080	2	2	2	$17,1$	2	1
	$2^4.3.5.17$	3	3	3	$17,5,2$	2	1
		5	5	5	$17,6,4,2,1$	3	1
		17	9	9	$17,8,7,6,5,4,3,2,1$	9	1
$PSL(2,17)$	2448	2	5	5	$11,4,3,2,1$	5	1
	$2^4.3^2.17$	3	5	5	$11,4,3,2,1$	5	1
		17	3	3	$11,2,1$	3	1
$PSL(2,19)$	3420	2	2	2	$12,3$	2	1
	$2^2.3^2.5.19$	3	5	5	$12,4,3,2,1$	5	1
		5	3	3	$12,4,2$	3	1
		19	3	3	$12,2,1$	3	1
$PSL(2,23)$	6072	2	4	4	$14,5,3,1$	3	1
	$2^3.3.11.23$	3	3	3	$14,4,1$	2	1
		11	6	6	$14,5,4,3,2,1$	6	1
		23	3	3	$14,2,1$	3	1
$PSL(2,25)$	7800	2	4	4	$15,5,3,1$	3	1
	$2^3.3.5^2.13$	3	3	3	$15,4,1$	2	1
		5	3	3	$15,2,1$	3	1
		13	7	7	$15,6,5,4,3,2,1$	7	1
$PSL(2,27)$	9828	2	2	2	$16,4$	2	1
	$2^2.3^3.7.13$	3	3	3	$16,2,1$	3	1
		7	4	4	$16,6,4,2$	4	1
		13	7	7	$16,6,5,4,3,2,1$	7	1

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